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Note :-

1) If $f(x)$ is continuous in closed interval $[a,b]$, then bisection method is not applicable.
2) If $f(x)$ is continuous in a closed interval $[a,b]$ and does not cut the $x$-axis, then $f(x)$ does not have a real root.
3) Bisection method gives the real root of $f(x)=0$
4) Bisection method is also called BOLZANO METHOD.

# Order of Convergence of iterative methods.

Convergence of an iterative method is defined as the order at which the error b/w any successive approximations to the root decreases.
An iterative method is said to have 'k' rate of order of convergence if 'k' is the largest value for which

$$\lim_{n\to\infty}\left|\frac{\varepsilon_{n+1}}{\varepsilon_n^k}\right| \leq C$$

where $C$ is a non-zero and finite number which is called asymptotic error constant and is dependent on the derivatives of $f(x)$ at an approximation of value of '$x$' where $\varepsilon_n$ & $\varepsilon_{n+1}$ are the errors in $n^{th}$ and $(n+1)^{th}$ approximations to the root respectively.

# Order of Convergence of Bisection Method

For Each iteration in Bisection method, the interval in which the root lies, is divided into half interval. If the middle points of successive intervals are taken as approximations to the root, one half of the current interval is the upper bound for the error; therefore

$$e_{n+1} = \frac{1}{2} e_n$$

$$\Rightarrow e_{n+1} \propto e_n$$

i.e., $e_{n+1}$ is directly proportional to $e_n$.
Thus the Bisection method is convergent linearly.

# To prove that Bisection Method is always convergent.

**Proof:** Let $[p_n, q_n]$ be the interval at $n^{th}$ iteration of Bisection method in which a root of the equation $f(x)=0$ lies. Let $x_n$ be the $n^{th}$ approximation to the root.

$\Rightarrow x_1 =$ first approximation

$$= \frac{p_1 + q_1}{2}$$

$$\Rightarrow p_1 < x_1 < q_1$$

Now we take that the roots lies either in $[p_1, x_1]$ or in $[x_1, q_1]$

$\therefore$ Either $[p_2, q_2] = [p_1, x_1]$ or $[p_2, x_2] = [x_1, q_1]$

$\Rightarrow p_2 = p_1 ; q_2 = x_1$ or $p_2 = x_1 , q_2 = q_1$

$\Rightarrow p_1 \leq p_2 , q_2 \leq q_1$

Again $x_2$ = second approximation.

$= \frac{p_2 + q_2}{2}$

$\Rightarrow p_2 < x_2 < q_2$

Similarly, we get $x_n = \frac{p_n + q_n}{2}$

$\Rightarrow p_n < x_n < q_n$

Now, we get two sequences

$p_1 \leq p_2 \leq p_3 \leq \dots \leq p_n$

$q_1 \geq q_2 \geq q_3 \geq \dots \geq q_n$

$\therefore$ The sequence $\langle p_n \rangle$ is non-decreasing & bounded by 'b' and sequence $\langle q_n \rangle$ is ~~non-incr~~ non-increasing bounded by 'a'.

Hence, the sequence converge.

Let $\lim_{n \to \infty} p_n = p$ and $\lim_{n \to \infty} q_n = q$

Since, the length of the interval decreases at each iteration

$\Rightarrow \lim_{n \to \infty} (q_n - p_n) = 0$

$\Rightarrow \lim_{n \to \infty} q_n - \lim_{n \to \infty} p_n = 0$

$\Rightarrow q - p = 0$

$\Rightarrow q = p$

But since $p_n < x_n < q_n$

$\Rightarrow \lim_{n \to \infty} p_n < \lim_{n \to \infty} x_n < \lim_{n \to \infty} q_n$

$\Rightarrow p < \lim_{n\to\infty} x_n < q$

$\Rightarrow \lim_{n\to\infty} x_n = p = q$ —— $(\because p = q)$ — (1)

Also since a root lies in $[p_n, q_n]$
we have $f(p_n) f(q_n) < 0$

$\Rightarrow \lim_{n\to\infty} \{f(p_n) f(q_n)\} \leq 0$

$\Rightarrow f(p) f(q) \leq 0$

$\Rightarrow \{f(p)\}^2 \leq 0$ —— $(\because p = q)$ — (2)

But square of any number is positive.

$\therefore f(p) = 0$

$\Rightarrow p$ is root of $f(x) = 0$

From eqn (1) & (2) we can say that the sequence $\langle x_n \rangle$ converges neccessarily to a root of the equation $f(x) = 0$

Proved